TRANSVERSE ELECTRIC SCATTERING ON INHOMOGENEOUS OBJECTS: SINGULAR INTEGRAL EQUATION, SYMBOL OF THE OPERATOR, AND MATRIX ELEMENTS

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Abstract. This is a companion report for the paper "Transverse electric scattering on inhomogeneous objects: Spectrum of integral operator and preconditioning" by the present authors [6]. In this report we formulate the two-dimensional transverse electric scattering problem as a standard singular integral equation, derive the symbol of the integral operator for Hölder-continuous contrasts, and calculate the elements of the system matrix obtained after discretization via the mid-point rule.

Key words. Domain integral equation, singular integral operators, electromagnetism, TE scattering, symbol of operators

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1. Derivation of the TE singular integral equation. Although some of the material presented here may be new, in particular, the explicit expression for the operator symbol, the methods and techniques are standard and can be found in [4, 3, 2].

Starting from the frequency-domain Maxwell's equations [6] with the time convention $e^{-i\omega t}$ we arrive at the following set of equations for the scattered fields with the induced currents as sources:

$$\begin{bmatrix} -i\omega\varepsilon_{\rm b} & 0 & -\partial_2 \\ 0 & -i\omega\varepsilon_{\rm b} & \partial_1 \\ -\partial_2 & \partial_1 & -i\omega\mu_{\rm b} \end{bmatrix} \begin{bmatrix} E_1^{\rm sc} \\ E_2^{\rm sc} \\ H_3^{\rm sc} \end{bmatrix} = \begin{bmatrix} -J_1^{\rm ind} \\ -J_2^{\rm ind} \\ -K_3^{\rm ind} \end{bmatrix}, \tag{1.1}$$

$$\begin{bmatrix} -i\omega\mu_{\rm b} & 0 & \partial_2 \\ 0 & -i\omega\mu_{\rm b} & -\partial_1 \\ \partial_2 & -\partial_1 & -i\omega\varepsilon_{\rm b} \end{bmatrix} \begin{bmatrix} H_1^{\rm sc} \\ H_2^{\rm sc} \\ E_3^{\rm sc} \end{bmatrix} = \begin{bmatrix} -K_1^{\rm ind} \\ -K_2^{\rm ind} \\ -J_3^{\rm ind} \end{bmatrix}, \tag{1.2}$$

where

$$\begin{aligned} J_k^{\rm ind}(\mathbf{x},\omega) &= -i\omega\left[\varepsilon(\mathbf{x},\omega)-\varepsilon_{\rm b}\right]E_k(\mathbf{x},\omega), \quad k=1,2,3; \\ K_m^{\rm ind}(\mathbf{x},\omega) &= -i\omega\left[\mu(\mathbf{x},\omega)-\mu_{\rm b}\right]H_m(\mathbf{x},\omega), \quad m=1,2,3. \end{aligned} \tag{1.3}$$

Equation (1.1) describes the TE case while (1.2) the TM case and is the dual of (1.1). The two-dimensional Fourier transform of (1.1) and (1.2) with respect to coordinates x_1 and x_2 takes us from the $(\mathbf{x},\omega)$ domain to the $(\mathbf{k},\omega)$ domain, whereby $\partial_n \to -ik_n$, $n=1,2$. In this way we arrive at the following linear algebraic problem of the form $\mathbb{A}\tilde{\mathbf{F}} = -\tilde{\mathbf{S}}$ for the TE case:

$$\begin{bmatrix} -i\omega\varepsilon_{\rm b} & 0 & ik_2 \\ 0 & -i\omega\varepsilon_{\rm b} & -ik_1 \\ ik_2 & -ik_1 & -i\omega\mu_{\rm b} \end{bmatrix} \begin{bmatrix} \tilde{E}_1^{\rm sc} \\ \tilde{E}_2^{\rm sc} \\ \tilde{H}_3^{\rm sc} \end{bmatrix} = - \begin{bmatrix} \tilde{J}_1^{\rm ind} \\ \tilde{J}_2^{\rm ind} \\ \tilde{K}_3^{\rm ind} \end{bmatrix}. \tag{1.4}$$

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Solving (1.4) via matrix inversion, we obtain the scattered fields in terms of the induced currents:

$$\begin{bmatrix} \tilde{E}_1^{\mathrm{sc}} \\ \tilde{E}_2^{\mathrm{sc}} \\ \tilde{H}_3^{\mathrm{sc}} \end{bmatrix} = \begin{bmatrix} \frac{k_1^2-\omega^2\varepsilon_{\mathrm{b}}\mu_{\mathrm{b}}}{i\omega\varepsilon_{\mathrm{b}}} & \frac{k_1k_2}{i\omega\varepsilon_{\mathrm{b}}} & ik_2 \\ \frac{k_1k_2}{i\omega\varepsilon_{\mathrm{b}}} & \frac{k_2^2-\omega^2\varepsilon_{\mathrm{b}}\mu_{\mathrm{b}}}{i\omega\varepsilon_{\mathrm{b}}} & -ik_1 \\ ik_2 & -ik_1 & i\omega\varepsilon_{\mathrm{b}} \end{bmatrix} \begin{bmatrix} \tilde{A}_1 \\ \tilde{A}_2 \\ \tilde{F}_3 \end{bmatrix}, \tag{1.5}$$

where we have introduced the vector potentials

$$\begin{aligned} \tilde{A}_k &= \frac{1}{k_1^2+k_2^2-\omega^2\varepsilon_{\mathrm{b}}\mu_{\mathrm{b}}}\tilde{J}_k^{\mathrm{ind}}, \quad k=1,2,3; \\ \tilde{F}_m &= \frac{1}{k_1^2+k_2^2-\omega^2\varepsilon_{\mathrm{b}}\mu_{\mathrm{b}}}\tilde{K}_m^{\mathrm{ind}}, \quad m=1,2,3. \end{aligned} \tag{1.6}$$

Transforming (1.5) back to the $(\mathbf{x},\omega)$ domain and recognizing the partial derivatives as $k_n \to i\partial_n$, $n=1,2$, we get

$$\begin{bmatrix} E_1^{\mathrm{sc}} \\ E_2^{\mathrm{sc}} \\ H_3^{\mathrm{sc}} \end{bmatrix} = \begin{bmatrix} -\frac{\partial_1^2}{i\omega\varepsilon_{\mathrm{b}}} - \frac{\omega^2\varepsilon_{\mathrm{b}}\mu_{\mathrm{b}}}{i\omega\varepsilon_{\mathrm{b}}} & -\frac{\partial_1\partial_2}{i\omega\varepsilon_{\mathrm{b}}} & -\partial_2 \\ -\frac{\partial_1\partial_2}{i\omega\varepsilon_{\mathrm{b}}} & -\frac{\partial_2^2}{i\omega\varepsilon_{\mathrm{b}}} - \frac{\omega^2\varepsilon_{\mathrm{b}}\mu_{\mathrm{b}}}{i\omega\varepsilon_{\mathrm{b}}} & \partial_1 \\ -\partial_2 & \partial_1 & i\omega\varepsilon_{\mathrm{b}} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ F_3 \end{bmatrix}. \tag{1.7}$$

The $(\mathbf{x},\omega)$–domain vector potentials are the spatial convolutions of the induced currents with the scalar Green's function, namely

$$\begin{aligned} A_k(\mathbf{x},\omega) &= \int_{\mathbf{x}'\in\mathbb{R}^2} g(\mathbf{x}-\mathbf{x}',\omega)J_k^{\mathrm{ind}}(\mathbf{x},\omega), \quad k=1,2,3; \\ F_m(\mathbf{x},\omega) &= \int_{\mathbf{x}'\in\mathbb{R}^2} g(\mathbf{x}-\mathbf{x}',\omega)K_m^{\mathrm{ind}}(\mathbf{x},\omega), \quad m=1,2,3. \end{aligned} \tag{1.8}$$

The Green's function $g(\mathbf{x},\omega)$ is the two-dimensional inverse Fourier transform of the expression $1/(k_1^2+k_2^2-\omega^2\varepsilon_{\mathrm{b}}\mu_{\mathrm{b}})$ appearing in (1.6). It is easy to verify that this Green's function satisfies the non-homogeneous Helmholtz equation with a line current (two-dimensional Dirac's delta function) as a source term, located at $\mathbf{x}'$ position. Since the scattered fields are supposed to satisfy the so-called radiation boundary condition (i.e. outgoing waves decaying at infinity), out of the two possible solutions of the said Helmholtz equation one choses

$$g(\mathbf{x}-\mathbf{x}',\omega) = \frac{i}{4}H_0^{(1)}(k_{\mathrm{b}}|\mathbf{x}-\mathbf{x}'|). \tag{1.9}$$

The other possible solution has the form of the Hankel function of the second kind, and is chosen when a different time convention is used, i.e. for the time-dependence of the form $e^{i\omega t}$.

Substituting the induced currents from (1.3) and expressing the scattered fields as $\mathbf{E}^{\mathrm{sc}} = \mathbf{E} - \mathbf{E}^{\mathrm{in}}$ and $\mathbf{H}^{\mathrm{sc}} = \mathbf{H} - \mathbf{H}^{\mathrm{in}}$, we arrive at the following integro-differential equations with the total fields as the fundamental unknown:

$$\begin{bmatrix} E_1^{\mathrm{in}} \\ E_2^{\mathrm{in}} \\ H_3^{\mathrm{in}} \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2 \\ H_3 \end{bmatrix} - \begin{bmatrix} k_{\mathrm{b}}^2+\partial_1^2 & \partial_1\partial_2 & -i\omega\mu_{\mathrm{b}}(-\partial_2) \\ \partial_2\partial_1 & k_{\mathrm{b}}^2+\partial_2^2 & -i\omega\mu_{\mathrm{b}}\partial_1 \\ -i\omega\varepsilon_{\mathrm{b}}(-\partial_2) & -i\omega\varepsilon_{\mathrm{b}}\partial_1 & k_{\mathrm{b}}^2 \end{bmatrix} \begin{bmatrix} g*(\chi_{\mathrm{e}}E_1) \\ g*(\chi_{\mathrm{e}}E_2) \\ g*(\chi_{\mathrm{m}}H_3) \end{bmatrix}, \tag{1.10}$$

where

$$\chi_{\rm e}(\mathbf{x},\omega) = \frac{\varepsilon(\mathbf{x},\omega)}{\varepsilon_{\rm b}} - 1, \tag{1.11}$$

$$\chi_{\rm m}(\mathbf{x},\omega) = \frac{\mu(\mathbf{x},\omega)}{\mu_{\rm b}} - 1, \tag{1.12}$$

are the normalized electric and magnetic contrast functions, respectively, and the star $(*)$ denotes the 2D convolution, for example

$$g * (\chi_{\rm e}E_1) = \int_{\mathbf{x}'\in\mathbb{R}^2} g(\mathbf{x}-\mathbf{x}',\omega)\chi_{\rm e}(\mathbf{x}')E_1(\mathbf{x}')\mathrm{d}\mathbf{x}'. \tag{1.13}$$

The system (1.10) contains nine scalar integro-differential operators. If the partial derivatives are carried out, then we arrive at nine "pure" integral operators whose kernels have different degrees of singularity. The weakly singular kernels result in compact operators, whereas the strongly singular kernels require some extra caution. The presence of the second-order partial derivatives in the upper left (2×2)-corner of the derivative matrix in (1.10) indicates that we have four scalar strongly singular kernels in the TE case. The corresponding operators are called singular integral operators.

To arrive at the standard form of the singular integral operator we first separate the domain of integration D in two sub-domains as $D = [D \setminus D(\epsilon)] \cup D(\epsilon)$ where $D(\epsilon)$ is a circular area around $\mathbf{x}$ with the radius ϵ. Let u_r be either of the electric field components E_1 or E_2, then the product between the operator matrix and the convolution vector in (1.10) can be written as

$$\begin{aligned} I_1 = &\lim_{\epsilon\to 0}\int_{\mathbf{x}'\in[D\setminus D(\epsilon)]} \partial_k\partial_r g(\mathbf{x}-\mathbf{x}')\chi_{\rm e}(\mathbf{x}')u_r(\mathbf{x}')\mathrm{d}\mathbf{x}' \\ &+ \lim_{\epsilon\to 0}\partial_k\int_{\mathbf{x}'\in D(\epsilon)} \partial_r g(\mathbf{x}-\mathbf{x}')\chi_{\rm e}(\mathbf{x}')u_r(\mathbf{x}')\mathrm{d}\mathbf{x}', \quad k,r=1,2. \end{aligned} \tag{1.14}$$

Here and in what follows we save some space by not showing the parametric dependence on ω. The first term in (1.14) is recognized as the principal value, while the second integral will be denoted by I_2. This second term incorporates the Green's function of (1.9). Utilizing asymptotic expansions for small arguments for the zero order Bessel and Neumann functions [1], namely

$$J_0(z) = 1 - \frac{z^2}{4} + O(z^4), \tag{1.15}$$

$$N_0(z) = \frac{2}{\pi}\gamma J_0(z) + \frac{2}{\pi}J_0(z)\ln\frac{z}{2} + O(z^2), \tag{1.16}$$

we replace the Hankel function in Green's function by its asymptotic expansion based on (1.15) and (1.16), arriving at

$$\begin{aligned} g(\mathbf{x}-\mathbf{x}') &= -\frac{\gamma}{2\pi} + \frac{i}{4} - \frac{1}{2\pi}\ln\frac{k_{\rm b}|\mathbf{x}-\mathbf{x}'|}{2} + O(|\mathbf{x}-\mathbf{x}'|^2) \\ &= g_0(\mathbf{x}-\mathbf{x}') + g_1(\mathbf{x}-\mathbf{x}'), \end{aligned} \tag{1.17}$$

where we have defined

$$g_0(\mathbf{x}-\mathbf{x}') = g(\mathbf{x}-\mathbf{x}') - \left[-\frac{1}{2\pi}\ln\frac{k_{\rm b}|\mathbf{x}-\mathbf{x}'|}{2}\right], \tag{1.18}$$

and

$$g_1(\mathbf{x}-\mathbf{x}') = -\frac{1}{2\pi}\ln\frac{k_{\mathrm{b}}|\mathbf{x}-\mathbf{x}'|}{2}. \tag{1.19}$$

The function $g_0(\mathbf{x}-\mathbf{x}')$ is not singular and its contribution (even after differentiation) in I_2 will vanish in the limit $\epsilon \to 0$. The nonzero contribution to I_2 comes from $g_1(\mathbf{x}-\mathbf{x}')$. Taking the first partial derivative ∂_r of $g_1(\mathbf{x}-\mathbf{x}')$, we get

$$\partial_r g_1(\mathbf{x}-\mathbf{x}') = \frac{1}{2\pi}\frac{-\Theta_r}{|\mathbf{x}-\mathbf{x}'|}, \tag{1.20}$$

where $\Theta_r = (x_r - x'_r)/|\mathbf{x}-\mathbf{x}'|$. A singularity of order one has appeared, which is still a weak singularity (the order of the singularity is smaller than the dimension of the manifold, which is two-dimensional in the present problem). Applying the second partial derivative ∂_k we obtain a strong second-order singularity:

$$\partial_k\partial_r g_1(\mathbf{x}-\mathbf{x}') = \frac{1}{2\pi}\partial_k\frac{-\Theta_r}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{2\pi}\frac{1}{|\mathbf{x}-\mathbf{x}'|^2}\left(2\Theta_k\Theta_r - \delta_{kr}\right). \tag{1.21}$$

Thus, omitting the g_0 part (since it disappears in the limit) we re-write the second term of (1.14) as

$$\begin{aligned}&\lim_{\epsilon\to 0}\partial_k\int_{\mathbf{x}'\in D(\epsilon)}\partial_r g(\mathbf{x}-\mathbf{x}')\chi_{\mathrm{e}}(\mathbf{x}')u_r(\mathbf{x}')\mathrm{d}\mathbf{x}'\\ &= \lim_{\epsilon\to 0}\int_{\mathbf{x}'\in D(\epsilon)}\frac{1}{2\pi}\partial_k\frac{-\Theta_r}{|\mathbf{x}-\mathbf{x}'|}\chi_{\mathrm{e}}(\mathbf{x}')u_r(\mathbf{x}')\mathrm{d}\mathbf{x}'.\end{aligned} \tag{1.22}$$

By adding and subtracting $\chi_{\mathrm{e}}(\mathbf{x})u_r(\mathbf{x})$ inside the integral of (1.22), we get

$$\begin{aligned}&\lim_{\epsilon\to 0}\partial_k\int_{\mathbf{x}'\in D(\epsilon)}\partial_r g(\mathbf{x}-\mathbf{x}')\chi_{\mathrm{e}}(\mathbf{x}')u_r(\mathbf{x}')\mathrm{d}\mathbf{x}'\\ &= \frac{1}{2\pi}\lim_{\epsilon\to 0}\int_{\mathbf{x}'\in D(\epsilon)}\partial_k\frac{-\Theta_r}{|\mathbf{x}-\mathbf{x}'|}\left[\chi_{\mathrm{e}}(\mathbf{x}')u_r(\mathbf{x}') - \chi_{\mathrm{e}}(\mathbf{x})u_r(\mathbf{x})\right]\mathrm{d}\mathbf{x}'\\ &+ \frac{1}{2\pi}\lim_{\epsilon\to 0}\chi_{\mathrm{e}}(\mathbf{x})u_r(\mathbf{x})\int_{\mathbf{x}'\in D(\epsilon)}\partial_k\frac{-\Theta_r}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}\mathbf{x}'.\end{aligned} \tag{1.23}$$

Assuming the Hölder continuity of the function $\chi_{\mathrm{e}}(\mathbf{x})u_r(\mathbf{x})$, i.e., assuming that there exist $\alpha, C > 0$, such that for all $\mathbf{x},\mathbf{x}' \in \mathbb{R}^2$,

$$|\chi_{\mathrm{e}}(\mathbf{x})u_r(\mathbf{x}) - \chi_{\mathrm{e}}(\mathbf{x}')u_r(\mathbf{x}')| \leq C|\mathbf{x}-\mathbf{x}'|^{\alpha}, \tag{1.24}$$

we effectively lower the order of singularity in the first integral in the RHS of (1.23). Hence, in the limit $\epsilon \to 0$ this term is zero. Interchanging ∂_k with $-\partial'_k$ and applying the 2D divergence theorem in the remaining term of (1.23), we obtain

$$\begin{aligned}&\lim_{\epsilon\to 0}\partial_k\int_{\mathbf{x}'\in D(\epsilon)}\partial_r g(\mathbf{x}-\mathbf{x}')\chi_{\mathrm{e}}(\mathbf{x}')u_r(\mathbf{x}')\mathrm{d}\mathbf{x}'\\ &= \frac{1}{2\pi}\chi_{\mathrm{e}}(\mathbf{x})u_r(\mathbf{x})\lim_{\epsilon\to 0}\oint_{\mathbf{x}'\in\partial D(\epsilon)}-\nu_k\frac{-\Theta_r}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}\mathbf{x}',\end{aligned} \tag{1.25}$$

with $\nu_k = -\Theta_k$ being the projection of the normal unit vector of the polar coordinate system on the x_k axis. Finally, we arrive at the result

$$\begin{aligned}
&\lim_{\epsilon\to 0}\partial_k \int_{\mathbf{x}'\in D(\epsilon)} \partial_r g(\mathbf{x}-\mathbf{x}')\chi_{\mathrm{e}}(\mathbf{x}')u_r(\mathbf{x}')\mathrm{d}\mathbf{x}' \\
&= \frac{1}{2\pi}\chi_{\mathrm{e}}(\mathbf{x})u_r(\mathbf{x}) \lim_{\epsilon\to 0}\oint_{\mathbf{x}'\in\partial D(\epsilon)} -\frac{\Theta_k\Theta_r}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}\mathbf{x}' \\
&= -\frac{1}{2\pi}\chi_{\mathrm{e}}(\mathbf{x})u_r(\mathbf{x}) \lim_{\epsilon\to 0}\oint_{\mathbf{x}'\in\partial D(1)} \Theta_k\Theta_r\mathrm{d}\mathbf{x}' \\
&= -\frac{1}{2}\chi_{\mathrm{e}}(\mathbf{x})u_r(\mathbf{x})\delta_{rk},
\end{aligned} \tag{1.26}$$

where we notice that the last contour integral is over the unit circle. Hence, (1.14) can now be written as

$$\begin{aligned}
I_1 = &\, p.\,v. \int_{\mathbf{x}'\in D} \partial_k\partial_r g(\mathbf{x}-\mathbf{x}')\chi_{\mathrm{e}}(\mathbf{x}')u_r(\mathbf{x}')\mathrm{d}\mathbf{x}' \\
&- \frac{1}{2}\chi_{\mathrm{e}}(\mathbf{x})u_r(\mathbf{x})\delta_{rk}, \quad k,r = 1,2.
\end{aligned} \tag{1.27}$$

Now, we can represent (1.10) in the standard form

$$\begin{aligned}
\begin{bmatrix} E_1^{\mathrm{in}} \\ E_2^{\mathrm{in}} \\ H_3^{\mathrm{in}} \end{bmatrix} = \begin{bmatrix} S & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ H_3 \end{bmatrix} &+ p.\,v. \begin{bmatrix} G_{11} & G_{12} & 0 \\ G_{21} & G_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} * \begin{bmatrix} X_{\mathrm{e}}E_1 \\ X_{\mathrm{e}}E_2 \\ X_{\mathrm{m}}H_3 \end{bmatrix} \\
&+ \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} * \begin{bmatrix} X_{\mathrm{e}}E_1 \\ X_{\mathrm{e}}E_2 \\ X_{\mathrm{m}}H_3 \end{bmatrix},
\end{aligned} \tag{1.28}$$

with S denoting the operator of (pointwise) multiplication with the function $s(\mathbf{x}) = 1+1/2\chi_{\mathrm{e}}(\mathbf{x})$, and I – the identity operator. The kernels in the principal-value operator in (1.28) are easily recognized using (1.21) and (1.10), and are given by

$$G_{nm}(\mathbf{x}) = -\frac{1}{2\pi|\mathbf{x}|^2}\left[2\Theta_n\Theta_m - \delta_{nm}\right], \quad n,m = 1,2. \tag{1.29}$$

The kernels in the compact operator in (1.28) are easily obtained using (3.3), (3.4) and (1.10), and are given by

$$\begin{aligned}
K_{nm} = &\left[\frac{1}{2\pi|\mathbf{x}|^2} - \frac{ik_{\mathrm{b}}}{4|\mathbf{x}|}H_1^{(1)}(k_{\mathrm{b}}|\mathbf{x}|)\right]\left[2\Theta_n\Theta_m - \delta_{nm}\right] \\
&+ \frac{ik_{\mathrm{b}}^2}{4}H_0^{(1)}(k_{\mathrm{b}}|\mathbf{x}|)\left[\Theta_n\Theta_m - \delta_{nm}\right], \quad n,m = 1,2;
\end{aligned} \tag{1.30}$$

$$K_{13} = -\frac{\omega\mu_{\mathrm{b}}k_{\mathrm{b}}\Theta_2}{4}H_1^{(1)}(k_{\mathrm{b}}|\mathbf{x}|), \tag{1.31}$$

$$K_{31} = -\frac{\omega\varepsilon_{\mathrm{b}}k_{\mathrm{b}}\Theta_2}{4}H_1^{(1)}(k_{\mathrm{b}}|\mathbf{x}|), \tag{1.32}$$

$$K_{23} = \frac{\omega\mu_{\mathrm{b}}k_{\mathrm{b}}\Theta_1}{4}H_1^{(1)}(k_{\mathrm{b}}|\mathbf{x}|), \tag{1.33}$$

$$K_{32} = \frac{\omega\varepsilon_{\mathrm{b}}k_{\mathrm{b}}\Theta_1}{4}H_1^{(1)}(k_{\mathrm{b}}|\mathbf{x}|), \tag{1.34}$$

$$K_{33} = -\frac{ik_{\mathrm{b}}^2}{4}H_0^{(1)}(k_{\mathrm{b}}|\mathbf{x}|). \tag{1.35}$$

2. Derivation of the Symbol. Using operator notation, (1.28) can be written as

$$(A\mathbf{E})(\mathbf{x}) = \left(\mathbb{E} + \frac{1}{2}\mathbb{M}\right)\mathbf{E}(\mathbf{x}) + \mathbb{A}^{\mathrm{s}}\mathbb{M}\mathbf{E}(\mathbf{x}') + \mathbb{K}\mathbf{E}(\mathbf{x}'), \tag{2.1}$$

where $\mathbb{A}^{\mathrm{s}}$ is a (matrix) singular integral operator, $\mathbb{K}$ is a (matrix) compact integral operator, $\mathbb{E}$ is a (matrix) identity operator, and $\mathbb{M}(\mathbf{x}) = \chi_{\mathrm{e}}(\mathbf{x})\mathbb{E}_2$ is a (matrix) multiplication operator, where the lower-right element of $\mathbb{E}_2$ is set to zero. The singular term of (2.1) is of the form

$$(A^{\mathrm{s}}\mathbb{M}\mathbf{E})(\mathbf{x}) = \int_{\mathbf{x}'\in D} \frac{\mathbb{F}(\mathbf{\Theta})}{|\mathbf{x}-\mathbf{x}'|^2}\chi_{\mathrm{e}}(\mathbf{x}')\mathbf{E}(x')\mathrm{d}\mathbf{x}'. \tag{2.2}$$

According to (1.21), the characteristic (matrix) function $\mathbb{F}(\mathbf{\Theta})$ has the form

$$\mathbb{F}(\mathbf{\Theta}) = -\frac{1}{2\pi}[2\mathbb{Q}(\mathbf{x}-\mathbf{x}') - \mathbb{I}_2], \tag{2.3}$$

where $\mathbb{I}_2$ is the (3×3) identity matrix with the lower-right element set to zero, and the tensor $\mathbb{Q}$ is given by

$$\mathbb{Q}(\mathbf{x}-\mathbf{x}') = \begin{bmatrix} \frac{(x_1-x_1')^2}{|\mathbf{x}-\mathbf{x}'|^2} & \frac{(x_1-x_1')(x_2-x_2')}{|\mathbf{x}-\mathbf{x}'|^2} & 0 \\ \frac{(x_1-x_1')(x_2-x_2')}{|\mathbf{x}-\mathbf{x}'|^2} & \frac{(x_2-x_2')^2}{|\mathbf{x}-\mathbf{x}'|^2} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.4}$$

For $\chi_{\mathrm{e}}(\mathbf{x})$ Hölder-continuous on $\mathbb{R}^2$, the symbol of the compound singular integral operator of (2.1) can be computed as (see [4]):

$$\mathrm{Smb}(\mathbb{A}) = \mathbb{I} + \frac{1}{2}\chi_{\mathrm{e}}(\mathbf{x})\mathbb{I}_2 + \mathrm{Smb}(\mathbb{A}^{\mathrm{s}})\chi_{\mathrm{e}}(\mathbf{x}), \tag{2.5}$$

where $\mathbb{I}$ is the ordinary (3×3) identity matrix. Since the characteristics $\mathbb{F}$ depends only on $\mathbf{x}-\mathbf{x}'$, the symbol of the singular integral operator $\mathbb{A}^{\mathrm{s}}$ is the Fourier transform of its kernel $\mathbb{F}(\mathbf{\Theta})/|\mathbf{x}-\mathbf{x}'|^2$ with respect to the variable $\mathbf{y} = \mathbf{x}-\mathbf{x}'$, i.e. it is a $\mathbf{k}$-domain matrix-valued function $\tilde{\mathbb{F}}^{\mathrm{s}}(\mathbf{k})$. Computing this Fourier transform is a daunting task, and we shall use a shortcut proposed in [4].

Let A^{s} be a single component of our matrix-valued operator $\mathbb{A}^{\mathrm{s}}$ and let $\mathbf{\Phi}^{\mathrm{s}}(\mathbf{k})$ denote its symbol, which is one of the components of the matrix-valued symbol function $\tilde{\mathbb{F}}^{\mathrm{s}}(\mathbf{k})$ we are trying to compute. The symbol $\mathrm{Smb}(A^{\mathrm{s}}) = \mathbf{\Phi}^{\mathrm{s}}(\mathbf{k})$ of a scalar simple singular integral operator can be expanded in a series of 2D spherical functions of order p, that is, in Fourier series of sines and cosines [4], namely

$$\mathbf{\Phi}^{\mathrm{s}}(\tilde{\mathbf{\Theta}}) = \sum_{p=0}^{\infty}\left[\gamma_{2,p}a_p^{(1)}Y_{p,2}^{(1)}(\tilde{\mathbf{\Theta}}) + \gamma_{2,p}a_p^{(2)}Y_{p,2}^{(2)}(\tilde{\mathbf{\Theta}})\right], \tag{2.6}$$

where $\tilde{\mathbf{\Theta}} = \mathbf{k}/|\mathbf{k}|$ is the unit vector in the $\mathbf{k}$-domain, $Y_{p,2}^{(1)} = \sin(p\tilde{\phi})$, and $Y_{p,2}^{(2)} = \cos(p\tilde{\phi})$ is the basis of the expansion, $a_p^{(1)}$ and $a_p^{(2)}$ are the expansion coefficients, and $\tilde{\phi}$ is the directional angle of the unit vector $\tilde{\mathbf{\Theta}}$. In (2.6), $\gamma_{2,p} = \pi i^p\Gamma(p/2)/\Gamma((2+p)/2)$ [4]. Since any component $f(\mathbf{\Theta})$ of the characteristic matrix-valued function $\mathbb{F}(\mathbf{\Theta})$

given by (2.3) depend only on $\mathbf{\Theta}$, we can expand each of them in a Fourier series as well

$$f(\mathbf{\Theta}) = \sum_{p=1}^{\infty} \left[a_p^{(1)} \sin(p\phi) + a_p^{(2)} \cos(p\phi) \right], \tag{2.7}$$

where ϕ is the directional angle of the unit vector $\mathbf{\Theta} = (\mathbf{x}-\mathbf{x}')/|\mathbf{x}-\mathbf{x}'|$. It was shown in [4] that the expansion coefficients in (2.7) are the same with those of (2.6). Using (2.3) and (2.4), we see that the components of the characteristic $\mathbb{F}$ are

$$\mathbb{F}(\mathbf{\Theta}) = -\frac{1}{2\pi}[2\mathbb{Q}(\mathbf{x}-\mathbf{x}') - \mathbb{I}_2] = -\frac{1}{2\pi} \begin{bmatrix} 2\cos^2\phi - 1 & 2\cos\phi\sin\phi & 0 \\ 2\cos\phi\sin\phi & 2\sin^2\phi - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.8}$$

To calculate the expansion coefficients, we must equate each element $[\mathbb{F}(\mathbf{\Theta})]_{kr}$, $k, r = 1, 2$, from (2.8) with the series from (2.7). For example, for the element $[\mathbb{F}(\mathbf{\Theta})]_{11}$ we have

$$-\frac{1}{2\pi}[2\cos^2\phi - 1] = -\frac{1}{2\pi}\cos(2\phi) = \sum_{p=1}^{\infty} \left[a_p^{(1)} \sin(p\phi) + a_p^{(2)} \cos(p\phi) \right]. \tag{2.9}$$

From this expansion it is obvious that the only nonzero expansion coefficient is $a_2^{(2)} = -1/(2\pi)$, while the rest are all zero (i.e. $a_p^{(1)} = 0\ \forall p$ and $a_p^{(2)} = 0\ \forall p \neq 2$). The same procedure is followed for the rest of the components in (2.8). Then, we substitute the known expansion coefficients in (2.6) to get the elements $[\tilde{\mathbb{F}}^{\mathrm{s}}(\tilde{\mathbf{\Theta}})]_{kr}$, $k, r = 1, 2$. Continuing our example, the element $[\tilde{\mathbb{F}}^{\mathrm{s}}(\tilde{\mathbf{\Theta}})]_{11}$ is obtained through the substitutions $\gamma_{2,2} = \pi i^2 \Gamma(1)/\Gamma(2) = -\pi$, $Y_{2,2}^{(2)} = \cos(2\tilde{\phi})$ and $a_2^{(2)} = -1/(2\pi)$, and therefore $[\tilde{\mathbb{F}}^{\mathrm{s}}(\tilde{\mathbf{\Theta}})]_{11} = 1/2\cos(2\tilde{\phi}) = \cos^2(\tilde{\phi}) - 1/2$. Following the same procedure for all components, we finally get

$$\mathrm{Smb}(\mathbb{A}^{\mathrm{s}}) = \tilde{\mathbb{F}}^{\mathrm{s}}(\tilde{\mathbf{\Theta}}) = \begin{bmatrix} \cos^2\tilde{\phi} - 1/2 & \sin\tilde{\phi}\cos\tilde{\phi} & 0 \\ \sin\tilde{\phi}\cos\tilde{\phi} & \sin^2\tilde{\phi} - 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbb{Q}(\mathbf{k}) - \frac{1}{2}\mathbb{I}_2. \tag{2.10}$$

Substituting the result of (2.10) back in (2.5), we get the symbol of the complete operator as the following (3×3) matrix-valued function:

$$\mathrm{Smb}(\mathbb{A})(\mathbf{x}, \mathbf{k}) = \mathbb{I} + \chi_{\mathrm{e}}(\mathbf{x})\mathbb{Q}(\mathbf{k}). \tag{2.11}$$

3. Derivation of the algebraic system matrix. Equation (1.28) defines an algebraic system $Au = b$ for the numerical evaluation of the fields. To derive the matrix elements, its more convenient to use the equivalent integral form of (1.28).

The application of the matrix operator of (1.10) –the matrix that contains the partial derivatives– on Green's functions, give us the Green's tensor. In order to get the equivalent integral form of (1.28), we need to split the Green's tensor in two parts. The first part, denoted by $\mathbb{A}(\mathbf{x}-\mathbf{x}')$, corresponds to the second order or mixed derivatives only, i.e.

$$\mathbb{A}(\mathbf{x}-\mathbf{x}') = \left(k_{\mathrm{b}}^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \partial_1^2 & \partial_1\partial_2 & 0 \\ \partial_2\partial_1 & \partial_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) g(\mathbf{x}-\mathbf{x}'). \tag{3.1}$$

The second part, denoted by $\mathbb{B}(\mathbf{x}-\mathbf{x}')$, corresponds to the first order derivatives only, i.e.

$$\mathbb{B}(\mathbf{x}-\mathbf{x}') = \begin{bmatrix} 0 & 0 & \partial_2 \\ 0 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{bmatrix} g(\mathbf{x}-\mathbf{x}'). \tag{3.2}$$

To obtain the explicit relations for $\mathbb{A}(\mathbf{x}-\mathbf{x}')$ and $\mathbb{B}(\mathbf{x}-\mathbf{x}')$, we first calculate the first order derivative which is

$$\partial_r g(\mathbf{x}-\mathbf{x}') = -\frac{i}{4}k_\mathrm{b}\frac{x_r - x_r'}{|\mathbf{x}-\mathbf{x}'|}H_1^{(1)}(k_\mathrm{b}|\mathbf{x}-\mathbf{x}'|), \quad r = 1,2. \tag{3.3}$$

Then, the mixed derivative is

$$\begin{aligned}\partial_k\partial_r g(\mathbf{x}-\mathbf{x}') =& \frac{i}{4}\left[k_\mathrm{b}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\left(2\frac{x_r - x_r'}{|\mathbf{x}-\mathbf{x}'|}\frac{x_k - x_k'}{|\mathbf{x}-\mathbf{x}'|} - \delta_{kr}\right)H_1^{(1)}(k_\mathrm{b}|\mathbf{x}-\mathbf{x}'|)\right. \\ &\left. - k_\mathrm{b}^2\frac{x_r - x_r'}{|\mathbf{x}-\mathbf{x}'|}\frac{x_k - x_k'}{|\mathbf{x}-\mathbf{x}'|}H_0^{(1)}(k_\mathrm{b}|\mathbf{x}-\mathbf{x}'|)\right], \quad k,r = 1,2;\end{aligned} \tag{3.4}$$

where δ_{kr} is the Kronecker's delta. So, the $\mathbb{A}$ tensor is given by

$$\begin{aligned}\mathbb{A}(\mathbf{x}-\mathbf{x}') =& \frac{i}{4}k_\mathrm{b}\frac{1}{|\mathbf{x}-\mathbf{x}'|}H_1^{(1)}(k_\mathrm{b}|\mathbf{x}-\mathbf{x}'|)[2\mathbb{Q}(\mathbf{x}-\mathbf{x}') - \mathbb{I}_2] \\ &- \frac{i}{4}k_\mathrm{b}^2 H_0^{(1)}(k_\mathrm{b}|\mathbf{x}-\mathbf{x}'|)[\mathbb{Q}(\mathbf{x}-\mathbf{x}') - \mathbb{I}_2].\end{aligned} \tag{3.5}$$

The tensor $\mathbb{Q}$ was introduced in (2.4) while $\mathbb{I}_2$ is, as explained in Section 2, the (3×3) identity matrix with the lower-right element set to zero. The $\mathbb{B}$ tensor is obtained with the use of (3.3)

$$\mathbb{B}(\mathbf{x}-\mathbf{x}') = -\frac{i}{4}k_\mathrm{b}H_1^{(1)}(k_\mathrm{b}|\mathbf{x}-\mathbf{x}'|)\boldsymbol{\Theta}(\mathbf{x}-\mathbf{x}')\times, \tag{3.6}$$

where we have now introduced the $\boldsymbol{\Theta}\times$ tensor given by

$$\boldsymbol{\Theta}(\mathbf{x}-\mathbf{x}')\times = \begin{bmatrix} 0 & 0 & \frac{x_2 - x_2'}{|\mathbf{x}-\mathbf{x}'|} \\ 0 & 0 & -\frac{x_1 - x_1'}{|\mathbf{x}-\mathbf{x}'|} \\ -\frac{x_2 - x_2'}{|\mathbf{x}-\mathbf{x}'|} & \frac{x_1 - x_1'}{|\mathbf{x}-\mathbf{x}'|} & 0 \end{bmatrix}. \tag{3.7}$$

After this splitting, we can easily express (1.28) in the following integral form

$$\begin{aligned}\mathbf{E}^\mathrm{in}(\mathbf{x}) =& \left[1 + \frac{1}{2}\chi_\mathrm{e}(\mathbf{x})\right]\mathbf{E}(\mathbf{x}) \\ &- p.v.\int_{\mathbf{x}'\in D}\mathbb{A}(\mathbf{x}-\mathbf{x}')\chi_\mathrm{e}(\mathbf{x}')\mathbf{E}(\mathbf{x}')\mathrm{d}\mathbf{x}' \\ &- i\omega\mu_\mathrm{b}\int_{\mathbf{x}'\in D}\mathbb{B}(\mathbf{x}-\mathbf{x}')\chi_\mathrm{m}(\mathbf{x}')\mathbf{H}(\mathbf{x}')\mathrm{d}\mathbf{x}',\end{aligned} \tag{3.8}$$

$$\begin{aligned}\mathbf{H}^\mathrm{in}(\mathbf{x}) =& \mathbf{H}(\mathbf{x}) - \int_{\mathbf{x}'\in D}\mathbb{A}(\mathbf{x}-\mathbf{x}')\chi_\mathrm{m}(\mathbf{x}')\mathbf{H}(\mathbf{x}')\mathrm{d}\mathbf{x}' \\ &+ i\omega\varepsilon_\mathrm{b}\int_{\mathbf{x}'\in D}\mathbb{B}(\mathbf{x}-\mathbf{x}')\chi_\mathrm{e}(\mathbf{x}')\mathbf{E}(\mathbf{x}')\mathrm{d}\mathbf{x}',\end{aligned} \tag{3.9}$$

where the vectors $\mathbf{E}(\mathbf{x}) = [E_1(\mathbf{x}), E_2(\mathbf{x}), 0]^T$ and $\mathbf{H}(\mathbf{x}) = [0, 0, H_3(\mathbf{x})]^T$. Same applies for $\mathbf{E}^{\text{in}}(\mathbf{x})$ and $\mathbf{H}^{\text{in}}(\mathbf{x})$.

Equation (3.8) can be rewritten as

$$\begin{aligned}\mathbf{E}^{\text{in}}(\mathbf{x}) = & \left[1 + \frac{1}{2}\chi_{\text{e}}(\mathbf{x})\right]\mathbf{E}(\mathbf{x}) \\ & - p.v. \int_{\mathbf{x}' \in D_{\text{s}}} \mathbb{A}(\mathbf{x} - \mathbf{x}')\chi_{\text{e}}(\mathbf{x}')\mathbf{E}(\mathbf{x}')\text{d}\mathbf{x}' \\ & - \int_{\mathbf{x}' \in D \setminus D_{\text{s}}} \mathbb{A}(\mathbf{x} - \mathbf{x}')\chi_{\text{e}}(\mathbf{x}')\mathbf{E}(\mathbf{x}')\text{d}\mathbf{x}' \\ & - i\omega\mu_{\text{b}} \int_{\mathbf{x}' \in D_{\text{s}}} \mathbb{B}(\mathbf{x} - \mathbf{x}')\chi_{\text{m}}(\mathbf{x}')\mathbf{H}(\mathbf{x}')\text{d}\mathbf{x}' \\ & - i\omega\mu_{\text{b}} \int_{\mathbf{x}' \in D \setminus D_{\text{s}}} \mathbb{B}(\mathbf{x} - \mathbf{x}')\chi_{\text{m}}(\mathbf{x}')\mathbf{H}(\mathbf{x}')\text{d}\mathbf{x}', \end{aligned} \tag{3.10}$$

where we have separated the domain D into a domain $D \setminus D_{\text{s}}$ "free" of singularity, and a domain D_{s} which encloses the position vector $\mathbf{x}$ and hence the singularity. In this way we have three integrals in the usual sense and one in the sense of principal value. Applying a simple collocation technique with the mid–point rule, the usual sense integrals over the domain $D \setminus D_{\text{s}}$ would be given by

$$\begin{aligned} & - \int_{\mathbf{x}' \in D \setminus D_{\text{s}}} \mathbb{A}(\mathbf{x} - \mathbf{x}')\chi_{\text{e}}(\mathbf{x}')\mathbf{E}(\mathbf{x}')\text{d}\mathbf{x}' \\ & \approx - \sum_{\substack{m=1 \\ m \neq n}}^{N-1} \mathbb{A}(\mathbf{x}_n - \mathbf{x}_m)\chi_{\text{e}}(\mathbf{x}_m)\mathbf{E}(\mathbf{x}_m)S_m, \quad n = 1, 2, \ldots, N; \end{aligned} \tag{3.11}$$

$$\begin{aligned} & - i\omega\mu_{\text{b}} \int_{\mathbf{x}' \in D \setminus D_{\text{s}}} \mathbb{B}(\mathbf{x} - \mathbf{x}')\chi_{\text{m}}(\mathbf{x}')\mathbf{H}(\mathbf{x}')\text{d}\mathbf{x}' \\ & \approx -i\omega\mu_{\text{b}} \sum_{\substack{m=1 \\ m \neq n}}^{N-1} \mathbb{B}(\mathbf{x}_n - \mathbf{x}_m)\chi_{\text{m}}(\mathbf{x}_m)\mathbf{H}(\mathbf{x}_m)S_m, \quad n = 1, 2, \ldots, N; \end{aligned} \tag{3.12}$$

where $S_m = h^2$ is the surface of each elementary cell D_m of side h in the computational domain.

We now proceed to calculate the principal value integral in (3.10). We have

$$\begin{aligned} & p.v. \int_{\mathbf{x}' \in D_{\text{s}}} \mathbb{A}(\mathbf{x} - \mathbf{x}')\chi_{\text{e}}(\mathbf{x}')\mathbf{E}(\mathbf{x}')\text{d}\mathbf{x}' \\ & = \lim_{\epsilon \to 0} \int_{\mathbf{x}' \in D_{\text{s}} \setminus D(\epsilon)} \mathbb{A}(\mathbf{x} - \mathbf{x}')\chi_{\text{e}}(\mathbf{x}')\mathbf{E}(\mathbf{x}')\text{d}\mathbf{x}' \\ & \approx \chi_{\text{e}}(\mathbf{x}_n)\mathbf{E}(\mathbf{x}_n) \lim_{\epsilon \to 0} \int_{\mathbf{x}' \in D_{\text{s}} \setminus D(\epsilon)} \mathbb{A}(\mathbf{x} - \mathbf{x}')\text{d}\mathbf{x}', \end{aligned} \tag{3.13}$$

where the domain of integration in the last integral is over the singular cell but with an exception of a small circular neighborhood having radius ϵ. We further define the tensor $\mathbb{L}$ as

$$\mathbb{L} = \lim_{\epsilon \to 0} \int_{\mathbf{x}' \in D_{\text{s}} \setminus D(\epsilon)} \mathbb{A}(\mathbf{x} - \mathbf{x}')\text{d}\mathbf{x}'. \tag{3.14}$$

Now we recall the form that $\mathbb{A}(\mathbf{x}-\mathbf{x}')$ has from (3.5). It is convenient to rewrite (3.5) in terms of $\mathbb{Q}(\mathbf{x}-\mathbf{x}')$ and $\mathbb{I}_2$, namely

$$\begin{aligned}\mathbb{A}(\mathbf{x}-\mathbf{x}') =& \frac{i}{4}\left[2k_\mathrm{b}\frac{1}{|\mathbf{x}-\mathbf{x}'|}H_1^{(1)}(k_\mathrm{b}|\mathbf{x}-\mathbf{x}'|) - k_\mathrm{b}^2 H_0^{(1)}(k_\mathrm{b}|\mathbf{x}-\mathbf{x}'|)\right]\mathbb{Q}(\mathbf{x}-\mathbf{x}') \\ &+ \frac{i}{4}\left[k_\mathrm{b}^2 H_0^{(1)}(k_\mathrm{b}|\mathbf{x}-\mathbf{x}'|) - k_\mathrm{b}\frac{1}{|\mathbf{x}-\mathbf{x}'|}H_1^{(1)}(k_\mathrm{b}|\mathbf{x}-\mathbf{x}'|)\right]\mathbb{I}_2. \end{aligned} \tag{3.15}$$

To compute (3.14), we transform the singular cell D_s from rectangular shape to a circular disk having center at $\mathbf{x}$ and radius a_n. The transformed circular cell with the original one have the same surface $S = \pi a_n^2 = h^2$. Then, (3.15) is transformed into polar coordinates as

$$\begin{aligned}\mathbb{A}(\rho,\phi) =& \frac{i}{4}\left[2k_\mathrm{b}\frac{1}{\rho}H_1^{(1)}(k_\mathrm{b}\rho) - k_\mathrm{b}^2 H_0^{(1)}(k_\mathrm{b}\rho)\right]\mathbb{Q}(\phi) \\ &+ \frac{i}{4}\left[k_\mathrm{b}^2 H_0^{(1)}(k_\mathrm{b}\rho) - k_\mathrm{b}\frac{1}{\rho}H_1^{(1)}(k_\mathrm{b}\rho)\right]\mathbb{I}_2. \end{aligned} \tag{3.16}$$

Now, we calculate each element of the tensor $\mathbb{L}$ from

$$\begin{aligned}L_{pq} =& \frac{i}{4}\Bigg\{\lim_{\epsilon\to 0}\int_{\rho=\epsilon}^{a_m}\left[2k_\mathrm{b}\frac{1}{\rho}H_1^{(1)}(k_\mathrm{b}\rho) - k_\mathrm{b}^2 H_0^{(1)}(k_\mathrm{b}\rho)\right]\rho\mathrm{d}\rho\int_{\phi=0}^{2\pi}Q_{pq}(\phi)\mathrm{d}\phi \\ &+ \int_{\rho=\epsilon}^{a_m}\left[k_\mathrm{b}^2 H_0^{(1)}(k_\mathrm{b}\rho) - k_\mathrm{b}\frac{1}{\rho}H_1^{(1)}(k_\mathrm{b}\rho)\right]\rho\mathrm{d}\rho\int_{\phi=0}^{2\pi}\delta_{pq}\mathrm{d}\phi\Bigg\}, \quad p,q=1,2. \end{aligned} \tag{3.17}$$

The integration in (3.17) is carried out only three times since the tensor $\mathbb{Q}(\mathbf{x}-\mathbf{x}')$ is symmetric. The angular integral with integrand function $Q_{pq}(\phi)$ gives $\pi\delta_{pq}$, while the other with the integrand function δ_{pq} gives $2\pi\delta_{pq}$. The radial integrals are evaluated by changing variables $z = k_\mathrm{b}\rho$ and using the well known integral formulas for Bessel functions [5]

$$\int^z \zeta^{n+1}Z_n(\zeta)\mathrm{d}\zeta = z^{n+1}Z_{n+1}(z), \tag{3.18}$$

$$\int^z \zeta^{-n+1}Z_n(\zeta)\mathrm{d}\zeta = -z^{-n+1}Z_{n-1}(z). \tag{3.19}$$

The result is

$$L_{pq} = \delta_{pq}\left\{\frac{i\pi a_n}{4}k_\mathrm{b}H_1^{(1)}(k_\mathrm{b}a_n) - \frac{i\pi}{4}k_\mathrm{b}\lim_{\epsilon\to 0}\left[\epsilon H_1^{(1)}(k_\mathrm{b}\epsilon)\right]\right\}, \quad p,q=1,2. \tag{3.20}$$

The involved limit is easily calculated giving $-i2/(\pi k_\mathrm{b})$. Therefore, substituting the results for the tensor $\mathbb{L}$ back to (3.13)

$$\begin{aligned}&p.v.\int_{\mathbf{x}'\in D_\mathrm{s}}\mathbb{A}(\mathbf{x}-\mathbf{x}')\chi_\mathrm{e}(\mathbf{x}')\mathbf{E}(\mathbf{x}')\mathrm{d}\mathbf{x}' \\ &\approx \chi_\mathrm{e}(\mathbf{x}_n)\left[-\frac{1}{2}+\frac{i\pi a_n}{4}k_\mathrm{b}H_1^{(1)}(k_\mathrm{b}a_n)\right]\mathbf{E}(\mathbf{x}_n), \quad \mathbf{x}_n\in D_\mathrm{s}. \end{aligned} \tag{3.21}$$

What remains is the calculation of the integral involving $\mathbb{B}(\mathbf{x}-\mathbf{x}')$ over the domain D_{s} in (3.10). We have

$$\begin{aligned}&-i\omega\mu_{\mathrm{b}}\int_{\mathbf{x}'\in D_{\mathrm{s}}}\mathbb{B}(\mathbf{x}-\mathbf{x}')\chi_{\mathrm{m}}(\mathbf{x}')\mathbf{H}(\mathbf{x}')\mathrm{d}\mathbf{x}'\\ &\approx -i\omega\mu_{\mathrm{b}}\chi_{\mathrm{m}}(\mathbf{x}_n)\mathbf{H}(\mathbf{x}_n)\int_{\mathbf{x}'\in D_{\mathrm{s}}}\mathbb{B}(\mathbf{x}-\mathbf{x}')\mathrm{d}\mathbf{x}'.\end{aligned} \tag{3.22}$$

Utilizing (3.6) and (3.7), and expressing $\mathbb{B}(\mathbf{x}-\mathbf{x}')$ in polar coordinates, in the same manner as we did for the $\mathbb{L}$ tensor before, we arrive at an angular integration from 0 to 2π for each component Θ_{pq} of $\mathbf{\Theta}(\mathbf{x}-\mathbf{x}')\times$ tensor. This integration is zero for each component Θ_{pq}, and hence this term does not contribute.

Putting together the results of (3.11), (3.12) and (3.21), we arrive at the linear equations

$$\begin{aligned}&\left\{1-\chi_{\mathrm{e}}(\mathbf{x}_n)\left[\frac{i\pi k_{\mathrm{b}}h}{4\sqrt{\pi}}H_1^{(1)}(k_{\mathrm{b}}h/\sqrt{\pi})-1\right]\right\}\mathbf{E}(\mathbf{x}_n)\\ &-h^2\sum_{\substack{m=1\\ m\neq n}}^{N-1}\mathbb{A}(\mathbf{x}_n-\mathbf{x}_m)\chi_{\mathrm{e}}(\mathbf{x}_m)\mathbf{E}(\mathbf{x}_m)\\ &-i\omega\mu_{\mathrm{b}}h^2\sum_{\substack{m=1\\ m\neq n}}^{N-1}\mathbb{B}(\mathbf{x}_n-\mathbf{x}_m)\chi_{\mathrm{m}}(\mathbf{x}_m)\mathbf{H}(\mathbf{x}_m)=\mathbf{E}^{\mathrm{in}}(\mathbf{x}_n),\quad n=1,2,\dots,N.\end{aligned} \tag{3.23}$$

Following the same analysis for (3.9), we arrive at the linear equations

$$\begin{aligned}&\left\{1-\chi_{\mathrm{m}}(\mathbf{x}_n)\left[\frac{i\pi k_{\mathrm{b}}h}{2\sqrt{\pi}}H_1^{(1)}(k_{\mathrm{b}}h/\sqrt{\pi})-1\right]\right\}\mathbf{H}(\mathbf{x}_n)\\ &-h^2\sum_{\substack{m=1\\ m\neq n}}^{N-1}\mathbb{A}(\mathbf{x}_n-\mathbf{x}_m)\chi_{\mathrm{m}}(\mathbf{x}_m)\mathbf{H}(\mathbf{x}_m)\\ &+i\omega\varepsilon_{\mathrm{b}}h^2\sum_{\substack{m=1\\ m\neq n}}^{N-1}\mathbb{B}(\mathbf{x}_n-\mathbf{x}_m)\chi_{\mathrm{e}}(\mathbf{x}_m)\mathbf{E}(\mathbf{x}_m)=\mathbf{H}^{\mathrm{in}}(\mathbf{x}_n),\quad n=1,2,\dots,N.\end{aligned} \tag{3.24}$$

Equations (3.23) and (3.24) compose the linear system for the evaluation of the unknown total fields, with the following structure

$$\begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix}=\begin{bmatrix}b_1\\ b_2\\ b_3\end{bmatrix}. \tag{3.25}$$

In (3.25), $[u_1,u_2,u_3]^T=[E_1(\mathbf{x}_n),E_2(\mathbf{x}_n),H_3(\mathbf{x}_n)]^T$, $n=1,2,\dots,N$, are the unknown total field components on the grid, while $[b_1,b_2,b_3]^T=[E_1^{\mathrm{in}}(\mathbf{x}_n),E_2^{\mathrm{in}}(\mathbf{x}_n),H_3^{\mathrm{in}}(\mathbf{x}_n)]^T$ contains the grid values of the incident field. The elements of the system matrix in (3.25) are now easily recognized with the use of (3.23), (3.24), (3.5), (3.6) and (1.9).

The results are

$$[A_{\ell\ell}]_{nm} = -k_{\mathrm{b}}^2 h^2 \chi_{\mathrm{e}}(\mathbf{x}_m)\Bigg\{ \left[\frac{i}{2k_{\mathrm{b}}r_{nm}} H_1^{(1)}(k_{\mathrm{b}}r_{nm}) - \frac{i}{4}H_0^{(1)}(k_{\mathrm{b}}r_{nm})\right]\theta_{\ell,nm}\theta_{\ell,nm} + \left[\frac{i}{4}H_0^{(1)}(k_{\mathrm{b}}r_{nm}) - \frac{i}{4k_{\mathrm{b}}r_{nm}}H_1^{(1)}(k_{\mathrm{b}}r_{nm})\right]\delta_{\ell\ell}\Bigg\}, \quad \ell = 1,2, \quad m \neq n; \tag{3.26}$$

$$[A_{\ell\ell}]_{nn} = 1 + \left[1 - \frac{i\pi k_{\mathrm{b}}h}{4\sqrt{\pi}}H_1^{(1)}(k_{\mathrm{b}}h/\sqrt{\pi})\right]\chi_{\mathrm{e}}(\mathbf{x}_n), \quad \ell = 1,2; \tag{3.27}$$

$$[A_{\ell q}]_{nm} = - k_{\mathrm{b}}^2 h^2 \chi_{\mathrm{e}}(\mathbf{x}_m)\left[\frac{i}{2k_{\mathrm{b}}r_{nm}} H_1^{(1)}(k_{\mathrm{b}}r_{nm}) - \frac{i}{4}H_0^{(1)}(k_{\mathrm{b}}r_{nm})\right] \times \theta_{\ell,nm}\theta_{q,nm}(1-\delta_{nm}), \quad \ell,q = 1,2, \quad \ell \neq q; \tag{3.28}$$

$$[A_{13}]_{nm} = i\omega\mu_{\mathrm{b}}h^2\chi_{\mathrm{m}}(\mathbf{x}_m)\frac{ik_{\mathrm{b}}}{4}H_1^{(1)}(k_{\mathrm{b}}r_{nm})\theta_{2,nm}(1-\delta_{nm}); \tag{3.29}$$

$$[A_{32}]_{nm} = - i\omega\varepsilon_{\mathrm{b}}h^2\chi_{\mathrm{e}}(\mathbf{x}_m)\frac{ik_{\mathrm{b}}}{4}H_1^{(1)}(k_{\mathrm{b}}r_{nm})\theta_{1,nm}(1-\delta_{nm}); \tag{3.30}$$

$$[A_{31}]_{nm} = i\omega\varepsilon_{\mathrm{b}}h^2\chi_{\mathrm{e}}(\mathbf{x}_m)\frac{ik_{\mathrm{b}}}{4}H_1^{(1)}(k_{\mathrm{b}}r_{nm})\theta_{2,nm}(1-\delta_{nm}); \tag{3.31}$$

$$[A_{23}]_{nm} = - i\omega\mu_{\mathrm{b}}h^2\chi_{\mathrm{m}}(\mathbf{x}_m)\frac{ik_{\mathrm{b}}}{4}H_1^{(1)}(k_{\mathrm{b}}r_{nm})\theta_{1,nm}(1-\delta_{nm}); \tag{3.32}$$

$$[A_{33}]_{nm} = -k_{\mathrm{b}}^2 h^2 \chi_{\mathrm{m}}(\mathbf{x}_m)\frac{i}{4}H_0^{(1)}(k_{\mathrm{b}}r_{nm}), \quad m \neq n; \tag{3.33}$$

$$[A_{33}]_{nn} = 1 + \left[1 - \frac{i\pi k_{\mathrm{b}}h}{2\sqrt{\pi}}H_1^{(1)}(k_{\mathrm{b}}h/\sqrt{\pi})\right]\chi_{\mathrm{m}}(\mathbf{x}_n). \tag{3.34}$$

In (3.26)–(3.34) we have defined $r_{nm} = |\mathbf{x}_n - \mathbf{x}_m|$, $\theta_{\ell,nm} = (x_{\ell,n} - x_{\ell,m})/r_{nm}$, $\ell = 1,2$, $n,m = 1,\dots,N$, and $x_{\ell,n}$ denotes the Cartesian component of the 2D position vector $\mathbf{x}_n$, pointing at the nth node of the grid.

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